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Uniqueness for plane crack problems in dipolar gradient elasticity and in couple-stress elasticity

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Abstract

The present work deals with the uniqueness theorem for plane crack problems in solids characterized by dipolar gradient elasticity. The theory of gradient elasticity derives from considerations of microstructure in elastic continua [Mindlin, R.D., 1964. Micro-structure in linear elasticity. Arch. Ration. Mech. Anal. 16, 51–78] and is appropriate to model materials with periodic structure. According to this theory, the strain-energy density assumes the form of a positive-definite function of the strain (as in classical elasticity) and the second gradient of the displacement (additional term). Specific cases of the general theory employed here are the well-known theory of couple-stress elasticity and the recently popularized theory of strain-gradient elasticity. These cases are also treated in the present study. We consider an anisotropic material response of the cracked plane body, within the linear version of gradient elasticity, and conditions of plane-strain or anti-plane strain. It is emphasized that, for crack problems in general, a uniqueness theorem more extended than the standard Kirchhoff theorem is needed because of the singular behavior of the solutions at the crack tips. Such a theorem will necessarily impose certain restrictions on the behavior of the fields in the vicinity of crack tips. In standard elasticity, a theorem was indeed established by Knowles and Pucik [Knowles, J.K., Pucik, T.A., 1973. Uniqueness for plane crack problems in linear elastostatics. J. Elast. 3, 155–160], who showed that the necessary conditions for solution uniqueness are a bounded displacement field and a bounded body-force field. In our study, we show that the additional (to the two previous conditions) requirement of a bounded displacement-gradient field in the vicinity of the crack tips guarantees uniqueness within the general form of the theory of dipolar gradient elasticity. In the specific cases of couple-stress elasticity and pure strain-gradient elasticity, the additional requirement is less stringent. This only involves a bounded rotation field for the first case and a bounded strain field for the second case.

Keywords: Crack problems; Uniqueness; Microstructure; Generalized continuum theories; Dipolar stresses; Gradient elasticity; Couple-stress elasticity

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1. Introduction

The present work is concerned with establishing a uniqueness theorem for plane crack problems within the framework of the linear theory of dipolar gradient elasticity. This theory was introduced by Mindlin (1964) in an effort to model the mechanical behavior of solids with *microstructure*. A related and even more general theory was also introduced at the same time by Green and Rivlin (1964). The basic concept of these generalized continuum theories lies in the consideration of a medium containing elements or particles (called macro-media), which are in themselves deformable media. This behavior can easily be realized if such a macro-particle is viewed as a collection of smaller sub-particles (called micro-media). In this way, each particle of the continuum is endowed with an *internal* displacement field, which can be expanded as a power series in internal coordinate variables. Within the above context, the lowest-order theory (dipolar or grade-two theory) is the one obtained by retaining only the first (linear) term of the foregoing series. Also, because of the inherent dependence of the strain energy on gradients of certain fields-like the displacement gradient (form I in Mindlin, 1964), the strain (form II) or the rotation (couple-stress case)—the new material constants imply the presence of characteristic lengths in the material behavior. These lengths can be related with the size of microstructure. In this way *size effects* can be incorporated in the stress analysis, a fact giving Mindlin's approach an advantage over the classical theory. Typical cases of continua amenable to such an analysis are periodic material structures like those, e.g., of crystal lattices, crystallites of a polycrystal or grains of a granular material.

As is well-known, ideas underlying generalized theories were advanced a long time ago by Cauchy (1851) and Voigt (1887), but the subject was generalized (e.g. including inertia effects) and reached maturity only with the works of Mindlin and Green and Rivlin mentioned above. The Mindlin theory and related ideas (see e.g. Bleustein, 1967; Mindlin and Eshel, 1968; Germain, 1973) had already some successful applications during the sixties (see e.g. Weitsman, 1966; Day and Weitsman, 1966; Herrmann and Achenbach, 1968). More recently, this approach and related extensions for microstructured materials have been employed to analyze various problems in, among other areas, wave propagation (Vardoulakis and Georgiadis, 1997; Georgiadis et al., 2000; Georgiadis and Velgaki, 2003; Georgiadis et al., 2004), fracture (Zhang et al., 1998; Shi et al., 2000; Georgiadis, 2003), mechanics of defects (Lubarda and Markenskoff, 2000; Lubarda, 2003), stability (Papargyri-Beskou et al., 2003), and plasticity (see e.g. Fleck et al., 1994; Vardoulakis and Sulem, 1995; Wei and Hutchinson, 1997; Begley and Hutchinson, 1998; Fleck and Hutchinson, 1998). Efficient numerical techniques (Shu et al., 1999; Amanatidou and Aravas, 2002; Engel et al., 2002; Tsepoura et al., 2002) have also been developed. Results from the aforementioned studies suggest that the Mindlin approach allows for the emergence of interesting boundary layer effects and extends the range of applicability of classical continuum theories towards bridging the gap between them and atomic-lattice theories.

In the present study, the most common version of Mindlin's theory is employed, i.e. the so-called micro-homogeneous case (see Section 10 in Mindlin, 1964). According to this, on the one hand, each material particle has three degrees of freedom (the displacement components—just as in the classical theories) and the micro-density does not differ from the macro-density, but, on the other hand, the Euler–Cauchy principle assumes a form with *non-vanishing* couple-stress vector and the strain-energy density depends not only upon the strain (as in standard elasticity) but also upon the *second gradient* of the displacement. This case is different from the general Cosserat (or micropolar) theory that takes material particles with six independent degrees of freedom (three displacement components and three rotation components, the latter involving rotation of a micro-medium w.r.t. its surrounding medium) but, as explained in Section 2 below, includes as important special cases the couple-stress elasticity and the strain-gradient elasticity.

Such a continuum (also known in the literature as a restricted Mindlin continuum) is best described, in the framework of theory of small strains and displacements, by the following form of the first law of thermodynamics with respect to a Cartesian coordinate system $Ox_1x_2x_3$: $\rho\dot{\mathsf{E}} = \tau_{pq}\dot{\varepsilon}_{pq} + m_{pqr}\partial_p\partial_q\dot{u}_r$, where ρ is the mass density, E is the internal energy per unit mass, u_r is the displacement vector, ε_{pq} is the linear strain tensor,

$\tau_{pq} = \tau_{qp}$ is the monopolar stress tensor, $m_{pqr} = m_{qpr}$ is the dipolar stress tensor (a third-rank tensor), $\partial_p() \equiv \partial()/\partial x_p$, a superposed dot denotes time derivative, and the Latin indices span the range (1, 2, 3). The dipolar stress tensor follows from the notion of *dipolar forces*, which are anti-parallel forces acting between the micro-media contained in the continuum (see Section 2 and the original expositions by Green and Rivlin (1964) and Jaunzemis (1967)). Finally, the above form of the first law of thermodynamics can be viewed as a more accurate description of the material response than that provided by the standard theory (case of $\rho\dot{\mathsf{E}} = \tau_{pq}\dot{\varepsilon}_{pq}$), if one thinks of a series expansion for $\rho\dot{\mathsf{E}}$ containing higher-order gradients of the displacement gradient (or even of its symmetrical part, the strain). For example, the additional terms may become significant in the vicinity of stress-concentration points where the displacement gradient undergoes steep variations.

In the present study, the problem of solution uniqueness within the linear version of dipolar gradient elasticity is addressed for cracked bodies, by considering an anisotropic material response under plane-strain or anti-plane strain conditions. It is emphasized at this point that the standard Kirchhoff theorem *cannot* guarantee uniqueness of solution in the case of crack problems. This is because of the fact that solutions to such problems exhibit singularities in the vicinity of crack tips, and some of these singularities may not be in harmony with the assumption of smooth fields in Kirchhoff's theorem. However, imposing certain restrictions on the behavior of the solution near the crack tips, a uniqueness theorem can be established. In other words, to prove uniqueness, a statement (in integral form) of the Principle of Virtual Work is necessary, in which statement all integrals involved should be *convergent* (in the case of divergent integrals, the Principle of Virtual Work simply has no meaning). Therefore, a uniqueness theorem for crack problems should determine the necessary conditions on certain near-tip fields that render *all* integrals convergent in the pertinent weak (variational) form.

In the quest of uniqueness, the bounds that are set for certain fields near the crack edges are the so-called *edge conditions*. A practical use of such conditions is known in classical elasticity crack problems (see e.g. Freund, 1990; Georgiadis and Brock, 1994; Georgiadis and Rigatos, 1996), since this information greatly helps one to obtain exact solutions via the Wiener–Hopf technique by knowing in advance the asymptotic behavior of at least some of the field quantities (e.g. the displacement), which are to be evaluated. In addition, candidate solutions in the course of solving a particular problem can immediately be discarded if they do not concur with the edge conditions.

Within classical linear elasticity, Knowles and Pucik (1973) proved that the necessary conditions for solution uniqueness are a bounded displacement field and a bounded body-force field in the crack-tip region. In our study, by following closely their approach, we show that the additional (to the two previous conditions) requirement of a bounded displacement-gradient field in the vicinity of crack tips guarantees uniqueness within the dipolar gradient elasticity. It is worthwhile noticing that recently obtained solutions to plane crack problems in the framework of this theory (Shi et al., 2000; Georgiadis, 2003) satisfy indeed the requirement of a bounded displacement-gradient in the vicinity of crack tip, since the crack-face displacement varies in these solutions like $O((-x_1)^{3/2})$ as $x_1 \to -0$, w.r.t. a Cartesian coordinate system $Ox_1x_2x_3$ attached to the crack tip and with the crack situated along ($-\ell < x_1 < 0$, $x_2 = 0$), where ℓ is a fixed length. We note in passing that such a cusp-like closure of the crack faces was also found recently by Cleveringa et al. (2000) through the use of discrete dislocations around the crack tip.

Finally, after treating here the general case of dipolar gradient theory (which involves the entire field of displacement gradient), we also deal with the special cases of couple-stress theory and strain-gradient theory.

2. Fundamentals of dipolar gradient elasticity

Here, we briefly present the basic ideas and equations of elastostatic dipolar gradient theory of small strains and displacements. The point of departure is the following form of the strain-energy density W in a 3D continuum

$$W \equiv W(\varepsilon_{pq}, \kappa_{rpq}), \tag{1}$$

where a Cartesian coordinate system $Ox_1x_2x_3$ is utilized with the Latin indices spanning the range (1,2,3), $\varepsilon_{pq} = (1/2)\ (\partial_p u_q + \partial_q u_p)$ is the linear strain tensor, $\partial_p(\,) \equiv \partial(\,)/\partial x_p$, u_q is the displacement vector, and $\kappa_{rpq} = \partial_r \partial_p u_q$ is the second gradient of displacement. The rotation tensor $\omega_{pq} = (1/2)(\partial_p u_q - \partial_q u_p)$ will also be employed later. Indicial notation and the summation convention are used throughout. In what follows, we assume the existence of a *positive definite* function $W(\varepsilon_{pq}, \kappa_{rpq})$. From the above definitions, the properties $\varepsilon_{pq} = \varepsilon_{qp}$, $\kappa_{rpq} = \kappa_{prq}$ and $\omega_{pq} = -\omega_{qp}$ are obvious. Simpler versions of the theory can be derived by identifying κ_{rpq} with either the rotation gradient (couple-stress theory: $\kappa_{rpq} = \partial_r \omega_{pq}$) or the strain gradient (strain-gradient theory: $\kappa_{rpq} = \partial_r \varepsilon_{pq}$). Nevertheless, we deal first with the general case by taking the gradient of the *entire* displacement-gradient field. Later, we will briefly consider the special cases too.

Clearly, the form in (1) allows, as well, for non-linear constitutive behavior of the material. However, in this study we confine interest only to a *linear* constitutive behavior and consider the following quadratic form of the strain-energy density:

$$W = (1/2)c_{pqlm}\varepsilon_{pq}\varepsilon_{lm} + (1/2)d_{rpqjlm}\kappa_{rpq}\kappa_{jlm}, \tag{2}$$

where (c_{pqlm}, d_{rpqjlm}) are tensors of the material constants. Notice that in these tensors (which are of even rank) the number of independent components can be reduced to yield isotropic behavior. In the general case, (c_{pqlm}, d_{rpqjlm}) can be considered as continuously differentiable functions of position (case of non-homogeneous behavior). Finally, the positive definiteness of W sets the usual restrictions on the range of values of the material constants. Inequalities of this type are given, e.g., in Georgiadis and Velgaki (2003) for the isotropic couple-stress case and in Georgiadis et al. (2004) for the isotropic strain-gradient case.

We should also mention that a 'mixed' term of the type $f_{rpqjl}\ \kappa_{rpq}\varepsilon_{jl}$ (with f_{rpqjl} being another tensor of material constants), which could be included in the RHS of (2), has been omitted because such a term precludes isotropic material behavior. This is because f_{rpqjl} is of odd rank and inevitably will result in preferred directions in the material response. In addition, the recent analysis of Georgiadis et al. (2004) in a concrete problem of application of dipolar gradient elasticity (the theory involves a specific form of constitutive relations with four independent material constants and studies the propagation of Rayleigh waves) showed that this term has a rather little effect. Notice that considering only isotropic constitutive relations, and therefore omitting the aforementioned 'mixed' term, is the choice in the fundamental work by Mindlin and Eshel (1968) and also seems to be the case in most of the recent studies employing the gradient approach. Nevertheless, the basic lines of the present uniqueness considerations do not alter by the presence of this term.

Further, stresses can be defined in terms of W in the standard variational manner

$$\tau_{pq} \equiv \frac{\partial W}{\partial \varepsilon_{pq}}, \qquad m_{rpq} \equiv \frac{\partial W}{\partial \kappa_{rpq}}, \tag{3a, b}$$

where $\tau_{pq} = \tau_{qp}$ is the monopolar (or Cauchy in the nomenclature of Mindlin, 1964) stress tensor and $m_{rpq} = m_{prq}$ is the dipolar (or double) stress tensor. The latter third-rank tensor follows from the notion of multipolar forces, which are anti-parallel forces acting between the micro-media contained in the continuum with microstructure (see Fig. 1). As explained by Green and Rivlin (1964) and Jaunzemis (1967), the notion of multipolar forces arises from a series expansion of the mechanical power M containing higher-order velocity gradients, i.e. $\mathsf{M} = F_q\dot{u}_q + F_{pq}(\partial_p\dot{u}_q) + F_{rpq}(\partial_r\partial_p\dot{u}_q) + \cdots$, where F_q are the usual (monopolar) forces of classical continuum mechanics and $(F_{pq}, F_{rpq}, \ldots)$ are the multipolar (dipolar, quadrupolar, etc.) forces within the framework of generalized continuum mechanics.

In this way, the resultant force on an ensemble of sub-particles can be viewed as being decomposed into *external* and *internal* forces, the latter ones being self-equilibrating. However, these self-equilibrating forces produce *non-vanishing* stresses, the multipolar stresses. This means that an element along a section or at the surface may transmit, besides the usual force vector, a *couple* vector as well (i.e. the Euler–Cauchy stress

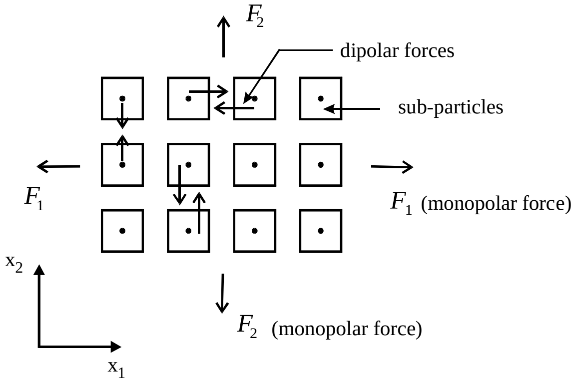

Fig. 1. A solid with microstructure: monopolar (external) and dipolar (internal) forces acting on an ensemble of sub-particles.

principle is augmented to include additional couple-tractions). Regarding the notation of the dipolar forces and stresses, the first index of the force indicates the orientation of the lever arm between the forces and the second one the orientation of the pair of forces itself. The same holds true for the last two indices of the dipolar stresses, whereas the first index denotes the orientation of the normal to the surface upon which the stress acts. Also, the dipolar forces F_{pq} have dimensions of [force][length]; their diagonal terms are double forces without moment and their off-diagonal terms are double forces with moment. In particular, the anti-symmetric part $F_{[pq]} = (1/2)(x_pF_q - x_qF_p)$ gives rise to couple-stresses. Finally, across a section with its outward unit normal in the positive direction, the force at the positive end of the lever arm is positive if it acts in the positive direction. 'Positive' refers to the positive sense of the coordinate axis parallel to the lever arm or force.

Then, Eqs. (2) and (3) provide the following constitutive relations:

$$\tau_{pq} = c_{pqlm}\varepsilon_{lm}, \quad m_{rpq} = d_{rpqjlm}\kappa_{jlm}. \tag{4a, b}$$

We also note that at certain points of the ensuing analysis, use will be made rather of the complementary strain-energy density $W^c \equiv W^c(\tau_{pq}, m_{rpq})$, which, however, is identical here with W due to the assumed linear constitutive behavior. Thus, we can also write $W \equiv W(\tau_{pq}, m_{rpq})$.

Next, the equations of equilibrium and the traction boundary conditions along a smooth boundary can be obtained either from Hamilton's principle (Mindlin, 1964) or from the momentum balance laws and their application on a material tetrahedron (Georgiadis et al., 2004). In particular, the issue of boundary conditions and their nature was elucidated by Bleustein (1967) in an important but not so widely known paper. These equations read (the first is the equation of equilibrium and the other two are the boundary conditions)

$$\partial_p(\tau_{pq} - \partial_r m_{rpq}) + f_q = 0, \tag{5}$$

$$n_p(\tau_{pq} - \partial_r m_{rpq}) - D_p(n_r m_{rpq}) + (D_j n_j) n_r n_p m_{rpq} = P_q, \tag{6}$$

$$n_r n_p m_{rpq} = R_q, \tag{7}$$

where f_q is the monopolar body force per unit volume, $D_p(\,) \equiv \partial_p() - n_p D(\,)$ is the surface gradient operator, $D(\,) \equiv n_r\partial_r(\,)$ is the normal gradient operator, n_p is the outward unit normal to the boundary, $P_q \equiv t_q^{(n)} + (D_r n_r) n_p T_{pq}^{(n)} - D_p T_{pq}^{(n)}$ is the auxiliary force traction, $R_q \equiv n_p T_{pq}^{(n)}$ is the auxiliary double force traction, $t_q^{(n)}$ is the *true* force surface traction, and $T_{pq}^{(n)}$ is the *true* double force surface traction. Examples of the latter tractions along the surface of a 2D half-space are given in Fig. 2. Notice that a dipolar body force

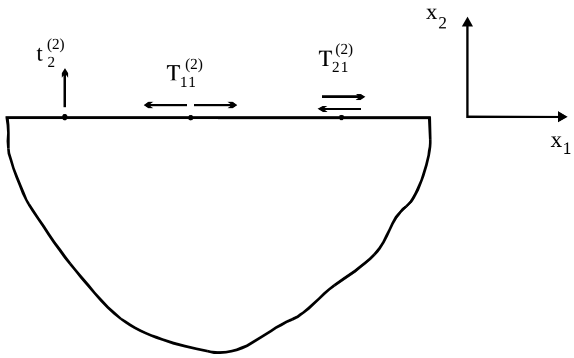

Fig. 2. Positively oriented true monopolar and dipolar tractions on the surface of a half-space.

field is omitted in the above equations since this case is a rather unrealistic possibility. This absence of double body forces can also be quoted in Mindlin's (1964) form I and, also, in Mindlin and Eshel (1968).

3. Uniqueness considerations in the general gradient theory

Consider a body occupying the *plane* domain $\widetilde{D}$ with a piecewise smooth boundary C. Either conditions of plane-strain or anti-plane strain are assumed to prevail. The body contains a single, internal, through the thickness, straight crack denoted by L. The treatment of a problem with many cracks follows also the lines of the present analysis. The crack faces are traction free and the loading is applied on C. Again, the treatment of a problem with non-zero tractions along the crack faces presents no additional difficulty. A continuous body-force field f_q may act on the body. As Fig. 3 depicts, a Cartesian coordinate system $Ox_1x_2x_3$ is placed on the body so that the crack consists of the points $L \equiv \{(x_1, x_2): -a \leqslant x_1 \leqslant a, x_2 = \pm 0\}$. Further,

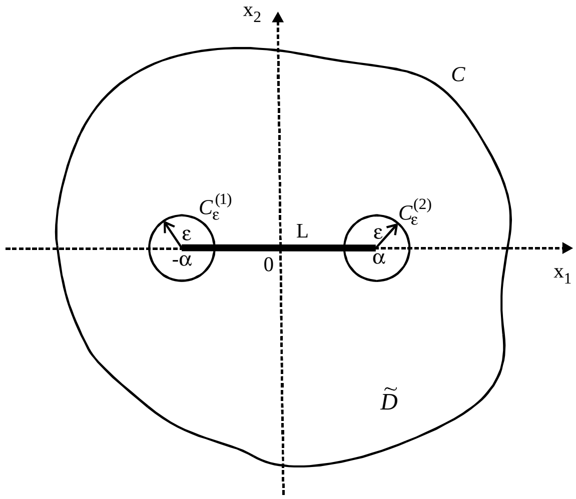

Fig. 3. A plane domain $\widetilde{D}$ having a boundary C and containing a finite-length crack L. The small circles $C_\varepsilon^{(1)}$ and $C_\varepsilon^{(2)}$ around the crack tips have radius ε.

we define two circles $C_\varepsilon^{(1)}$ and $C_\varepsilon^{(2)}$ around the crack tips $(-a,0)$ and $(a,0)$, respectively, and also write $C_\varepsilon \equiv C_\varepsilon^{(1)} + C_\varepsilon^{(2)}$. Each circle has a radius ε, which is positive and small enough to ensure that the circles are indeed within $\widetilde{D}$. We also define the following domains: $G_\varepsilon \equiv G_\varepsilon^{(1)} + G_\varepsilon^{(2)}$ is the domain that contains all points inside the circles but not lying on the crack L, $D_0 \equiv \widetilde{D} - L$ is the domain that contains those points of $\widetilde{D}$ not lying on L, $D_\varepsilon \equiv D_0 - G_\varepsilon$ is the domain that contains those points of D_0 which are outside the circles $C_\varepsilon^{(1)}$ and $C_\varepsilon^{(2)}$, and $D^+ \equiv \{(x_1,x_2) : (x_1,x_2) \in \widetilde{D} + C,\ x_2 \geqslant 0,\ (x_1 \pm a)^2 + x_2^2 \neq 0\}$ is the domain that contains those points of $\widetilde{D} + L$ lying on or above the x_1-axis with the exclusion of the two crack tips (similarly, $D^- \equiv \{(x_1,x_2) : (x_1,x_2) \in \widetilde{D} + C, x_2 \leqslant 0, (x_1 \pm a)^2 + x_2^2 \neq 0\}$). Finally, use will be made of the interior of L defined as $L_0 \equiv \{(x_1,x_2){:} -a < x_1 < a,\ x_2 = \pm 0\}$.

Of course, all forces and tractions act 'inside' the plane (x_1,x_2) and are independent upon x_3.

The *boundary value problem* for the cracked body is now stated as follows: Determine the solution $(u_q, \tau_{pq}, m_{rpq})$ to the problem described by Eqs. (4) and (5) holding on D_0, by the general conditions (6) and (7) holding along any boundary, and under given tractions P_q^* and R_q^* on the boundary C and stress-free boundary conditions on L_0.

More specifically, the latter conditions on L_0 are written in plane-strain as

$$\tau_{22} - \frac{\partial m_{122}}{\partial x_1} - \frac{\partial m_{222}}{\partial x_2} - \frac{\partial m_{212}}{\partial x_1} = 0, \quad \tau_{21} - \frac{\partial m_{121}}{\partial x_1} - \frac{\partial m_{221}}{\partial x_2} - \frac{\partial m_{211}}{\partial x_1} = 0, \tag{8a, b}$$

$$m_{222} = 0, \quad m_{221} = 0, \tag{9a, b}$$

and in anti-plane strain as

$$\tau_{23} - \frac{\partial m_{123}}{\partial x_1} - \frac{\partial m_{223}}{\partial x_2} - \frac{\partial m_{213}}{\partial x_1} = 0, \tag{10}$$

$$m_{223} = 0. \tag{11}$$

In particular, the LHS of Eqs. (8a), (8b) and (10) is the so-called *total* stress (Georgiadis, 2003), which is to be evaluated *ahead* of the crack tip in each of the cases of, respectively, mode-I, mode-II and mode-III fracture. As shown by Georgiadis (2003), the total stress ahead of the crack tip and the crack-face displacement are the quantities of main interest in each specific crack problem, since they enter in the expression of the energy release rate. Moreover, the total stress ahead of the crack tip can be related with the cleavage strength of the material.

Finally, in addition to the governing equations mentioned above, the solution to the boundary value problem should obey the appropriate *smoothness* conditions given below. These conditions stem from the requirement that the field quantities should be sufficiently smooth so that the governing equations of the problem are valid everywhere in the body:

$$u_q \in \mathbf{C}^1(D^+) \cap \mathbf{C}^1(D^-) \cap \mathbf{C}^4(D_0), \tag{12a}$$

$$\tau_{pq} \in \mathbf{C}(D^+) \cap \mathbf{C}(D^-) \cap \mathbf{C}^1(D_0), \tag{12b}$$

$$m_{rpq} \in \mathbf{C}^1(D^+) \cap \mathbf{C}^1(D^-) \cap \mathbf{C}^2(D_0). \tag{12c}$$

In considering uniqueness of the counterpart problem in classical elasticity, Knowles and Pucik (1973) showed that the necessary conditions to have a bounded strain-energy density (and, therefore, applicability of the Principle of Virtual Work and Kirchhoff's theorem) are a bounded displacement field and a bounded body-force field in the vicinity of crack tips. In the present study, we follow closely their approach and show that another one, the condition of bounded displacement-gradient, should supplement the latter conditions in order to render the following relations valid

$$\int\int_{D_0} W\,\mathrm{d}A < \infty, \tag{13}$$

$$2\int\int_{D_0} W\,\mathrm{d}A = \int\int_{D_0} f_q u_q\,\mathrm{d}A + \int_C P_q^* u_q\,\mathrm{d}s + \int_C R_q^* D u_q\,\mathrm{d}s \tag{14}$$

and thus be able to prove a uniqueness theorem for the crack problem of dipolar gradient elasticity stated before. In the above relations, dA is the infinitesimal element of area, ds is the infinitesimal element of arc length, and $D(\,)$ (not to be confused with any domain) is the normal gradient operator defined already in Section 2. Notice also that (14) derives from the Principle of Virtual Work (for the theory employed here) given in Mindlin (1964) and in Appendix A of the present paper. Equation (14) is but a generalization of the classical Clapeyron's theorem.

Indeed, once the relations (13) and (14) are established, uniqueness follows as in the standard Kirchhoff theorem, i.e. by considering two linearly independent solutions and proving through (14) that their *difference* (which is also a solution, a fact owing to the linearity of the problem) is equal to zero. In Appendix A, we provide the extension of Kirchhoff's theorem needed for dipolar gradient elasticity. Thus, in what follows, we will concentrate on determining the conditions that guarantee the validity of relations (13) and (14).

We start this task by proving the validity of the following equation

$$2\int\int_{D_\varepsilon} W\,\mathrm{d}A = \int\int_{D_\varepsilon} f_q u_q\,\mathrm{d}A + \int_C P_q^* u_q\,\mathrm{d}s + \int_C R_q^* D u_q\,\mathrm{d}s + \int_{C_\varepsilon} P_q u_q\,\mathrm{d}s + \int_{C_\varepsilon} R_q D u_q\,\mathrm{d}s, \tag{15}$$

where (P_q^*, R_q^*) are the given (known) tractions on the boundary C, whereas (P_q, R_q) are unknown tractions along C_ε. The latter tractions are related, of course, with the stresses through (6) and (7). To prove (15), we employ the divergence theorem and Eqs. (5)–(7). The proof runs as follows:

$$\begin{aligned}
&\int\int_{D_\varepsilon} f_q u_q\,\mathrm{d}A + \int_C P_q^* u_q\,\mathrm{d}s + \int_C R_q^* D u_q\,\mathrm{d}s + \int_{C_\varepsilon} P_q u_q\,\mathrm{d}s + \int_{C_\varepsilon} R_q D u_q\,\mathrm{d}s \\
&= -\int\int_{D_\varepsilon} \sigma_{pq,p} u_q dA + \int_{\partial D_\varepsilon} \left[n_p \sigma_{pq} - D_p(n_r m_{rpq}) + (D_j n_j) n_r n_p m_{rpq}\right] u_q\,\mathrm{d}s + \int_{\partial D_\varepsilon} n_r n_p m_{rpq} D u_q\,\mathrm{d}s \\
&= -\int\int_{D_\varepsilon} \sigma_{pq,p} u_q\,\mathrm{d}A + \int\int_{D_\varepsilon} \left(\sigma_{pq} u_q\right)_{,p} dA + \int_{\partial D_\varepsilon} n_r m_{rpq} u_{q,p}\,\mathrm{d}s \\
&= \int\int_{D_\varepsilon} \sigma_{pq} u_{q,p}\,\mathrm{d}A + \int_{\partial D_\varepsilon} n_r m_{rpq} u_{q,p}\,\mathrm{d}s \\
&= \int\int_{D_\varepsilon} \tau_{pq} u_{q,p}\,\mathrm{d}A - \int\int_{D_\varepsilon} m_{rpq,r} u_{q,p}\,\mathrm{d}A + \int\int_{D_\varepsilon} \left(m_{rpq} u_{q,p}\right)_{,r}\mathrm{d}A \\
&= \int\int_{D_\varepsilon} \tau_{pq}\varepsilon_{pq}\,\mathrm{d}A + \int\int_{D_\varepsilon} m_{rpq} u_{q,pr} dA = 2\int\int_{D_\varepsilon} W\,\mathrm{d}A,
\end{aligned} \tag{16}$$

where it is noticed that $(\,)_{,p} \equiv \partial_p(\,) \equiv \partial(\,)/\partial x_p$, $\sigma_{pq} \equiv \tau_{pq} - m_{rpq,r}$ is a stress-like quantity introduced for convenience in the computations, ∂D_ε denotes the boundary of the domain D_ε, and $W \equiv \int_0^{\varepsilon_{pq}} \tau_{pq}\,\mathrm{d}\varepsilon_{pq} + \int_0^{u_{q,pr}} m_{rpq}\,\mathrm{d}u_{q,pr} = (1/2)\tau_{pq}\varepsilon_{pq} + (1/2)m_{rpq}u_{q,pr}$.

Next, for all positive and sufficiently small numbers ε, we define a function $f(\varepsilon)$ by

$$f(\varepsilon) \equiv 2\int\int_{D_\varepsilon} W\,\mathrm{d}A - \int\int_{D_0} f_q u_q\,\mathrm{d}A - \int_C P_q^* u_q\,\mathrm{d}s - \int_C R_q^* D u_q\,\mathrm{d}s \tag{17}$$

aiming at showing eventually that $\lim_{\varepsilon\to+0} f(\varepsilon) = 0$ and, accordingly, that $\int\int_{D_0} W\,\mathrm{d}A \equiv lim_{\varepsilon\to+0}\int\int_{D_\varepsilon} W\,\mathrm{d}A < \infty$. Under the above definition, we can write (15) in the form

$$f(\varepsilon) = -\int\int_{G_\varepsilon} f_q u_q \,\mathrm{d}A + \int_{C_\varepsilon} P_q u_q \,\mathrm{d}s + \int_{C_\varepsilon} R_q D u_q \,\mathrm{d}s, \tag{18}$$

where we recall that $G_\varepsilon \equiv D_0 - D_\varepsilon$. Also, the derivative $f'(\varepsilon)$ will be employed in the ensuing analysis. From the definition in (17) and by performing a Leibnitz-type differentiation and taking into account the orientation of the outwardly-directed unit-vector normal to the contour C_ε (this vector n_p is 'inwards' w.r.t. D_ε—cf. Fig. 4), it can be deduced that

$$f'(\varepsilon) = -2\int_{C_\varepsilon} W \,\mathrm{d}s \tag{19}$$

which, in view of the positive definiteness of W, provides that $f'(\varepsilon) \leqslant 0$. The latter result shows that $f(\varepsilon)$ is a monotone *non-increasing* function. This will conveniently be employed in the sequel.

Further, working on the last two terms of the RHS of (18) and taking into account the general equations of boundary conditions in (6) and (7) and also considering which stress components do work and which are workless, one obtains

$$\int_{C_\varepsilon} P_q u_q \,\mathrm{d}s + \int_{C_\varepsilon} R_q D u_q \,\mathrm{d}s = \int_{C_\varepsilon} n_p \tau_{pq} u_q \,\mathrm{d}s + \int_{C_\varepsilon} n_r m_{rpq} u_{q,p} \,\mathrm{d}s. \tag{20}$$

Then, in light of (18) and (20) and the well-known triangle and Cauchy–Schwarz inequalities, we can have the following bound for $|f(\varepsilon)|$

$$\begin{aligned}
|f(\varepsilon)| &= \left| -\int\int_{G_\varepsilon} f_q u_q \,\mathrm{d}A + \int_{C_\varepsilon} P_q u_q \,\mathrm{d}s + \int_{C_\varepsilon} R_q D u_q \,\mathrm{d}s \right| \\
&\leqslant \left| -\int\int_{G_\varepsilon} f_q u_q \,\mathrm{d}A \right| + \left| \int_{C_\varepsilon} P_q u_q \,\mathrm{d}s + \int_{C_\varepsilon} R_q D u_q \,\mathrm{d}s \right| \\
&\leqslant \int\int_{G_\varepsilon} |f_q u_q| \,\mathrm{d}A + \int_{C_\varepsilon} |n_p \tau_{pq} u_q| \,\mathrm{d}s + \int_{C_\varepsilon} |n_r m_{rpq} u_{q,p}| \,\mathrm{d}s \\
&\leqslant \left(\int\int_{G_\varepsilon} |f_q|^2 \,\mathrm{d}A \right)^{1/2} \left(\int\int_{G_\varepsilon} |u_q|^2 \,\mathrm{d}A \right)^{1/2} + \left(\int_{C_\varepsilon} |n_p \tau_{pq}|^2 \,\mathrm{d}s \right)^{1/2} \left(\int_{C_\varepsilon} |u_q|^2 \,\mathrm{d}s \right)^{1/2} \\
&\quad + \left(\int_{C_\varepsilon} |n_r m_{rpq}|^2 \,\mathrm{d}s \right)^{1/2} \left(\int_{C_\varepsilon} |u_{q,p}|^2 \,\mathrm{d}s \right)^{1/2} \\
&\leqslant \left(\int\int_{G_\varepsilon} f_q f_q \,\mathrm{d}A \right)^{1/2} \left(\int\int_{G_\varepsilon} u_q u_q \,\mathrm{d}A \right)^{1/2} + \left(\int_{C_\varepsilon} \tau_{pq} \tau_{pq} \,\mathrm{d}s \right)^{1/2} \left(\int_{C_\varepsilon} u_q u_q \,\mathrm{d}s \right)^{1/2} \\
&\quad + \left(\int_{C_\varepsilon} m_{rpq} m_{rpq} \,\mathrm{d}s \right)^{1/2} \left(\int_{C_\varepsilon} u_{q,p} u_{q,p} \,\mathrm{d}s \right)^{1/2}.
\end{aligned} \tag{21}$$

Next, by considering bounded fields for the body force, displacement and displacement gradient everywhere in D_0, i.e. by considering the validity of the following inequalities (with (α, β, γ) being positive constants):

$$|f_q| \leqslant \alpha \quad \forall x_p \in D_0, \tag{22a}$$

$$|u_q| \leqslant \beta \quad \forall x_p \in D_0, \tag{22b}$$

$$|u_{q,p}| \leqslant \gamma \quad \forall x_p \in D_0, \tag{22c}$$

one obtains the following relations:

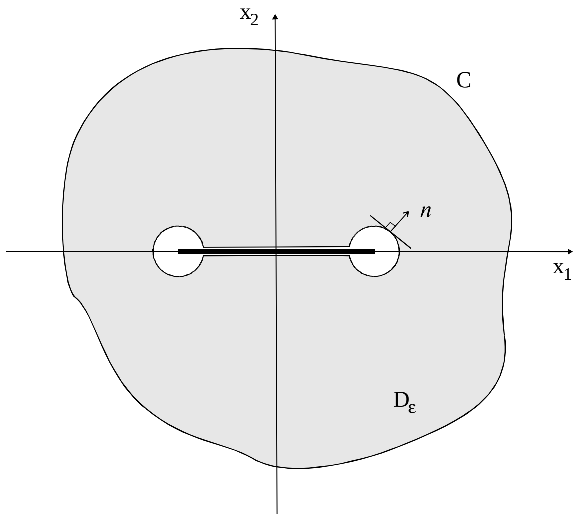

Fig. 4. The plane domain D_ε (shaded area) containing those points of $\widetilde{D}$ which are outside the circles $C_\varepsilon^{(1)}$ and $C_\varepsilon^{(2)}$ and, also, do not lie on the crack. The unit vector **n** is oriented 'inwards' with respect to D_ε.

$$\left(\int\int_{G_\varepsilon} f_q f_q \, \mathrm{d}A\right)^{1/2} \leqslant \alpha\left(\int\int_{G_\varepsilon} \mathrm{d}A\right)^{1/2} = \alpha\left(4\pi\varepsilon^2\right)^{1/2}, \tag{23}$$

$$\left(\int\int_{G_\varepsilon} u_q u_q \, \mathrm{d}A\right)^{1/2} \leqslant \beta\left(\int\int_{G_\varepsilon} \mathrm{d}A\right)^{1/2} = \beta\left(4\pi\varepsilon^2\right)^{1/2}, \tag{24}$$

$$\left(\int_{C_\varepsilon} u_q u_q \, \mathrm{d}s\right)^{1/2} \leqslant \beta\left(\int_{C_\varepsilon} \mathrm{d}s\right)^{1/2} = \beta(8\pi\varepsilon)^{1/2}, \tag{25}$$

$$\left(\int_{C_\varepsilon} u_{q,p} u_{q,p} \, \mathrm{d}s\right)^{1/2} \leqslant \gamma\left(\int_{C_\varepsilon} \mathrm{d}s\right)^{1/2} = \gamma(8\pi\varepsilon)^{1/2}. \tag{26}$$

Finally, by employing the latter relations in (21) we get the following bound for $|f(\varepsilon)|$:

$$| f(\varepsilon) | \leqslant 4\pi\varepsilon^2\alpha\beta + \beta(8\pi\varepsilon)^{1/2}\left(\int_{C_\varepsilon} \tau_{pq}\tau_{pq} \, \mathrm{d}s\right)^{1/2} + \gamma(8\pi\varepsilon)^{1/2}\left(\int_{C_\varepsilon} m_{rpq}m_{rpq} \, \mathrm{d}s\right)^{1/2}. \tag{27}$$

As will become clear soon, the inequalities in (22) are the necessary *edge conditions* that guarantee a unique solution to the crack boundary value problem.

At this point, it is advantageous to exploit the fact that the strain-energy density is not only positive definite but also a *quadratic* function in both τ_{pq} and m_{rpq}. Of course, the latter is true because we have assumed linear constitutive behavior (cf. Eq. (2)). Thus, there exist positive constants λ and μ such that

$$W(\tau_{pq}, m_{rpq}) \geqslant \lambda^2\tau_{pq}\tau_{pq} + \mu^2 m_{rpq}m_{rpq}. \tag{28}$$

Moreover, since obviously $\tau_{pq}\tau_{pq} \geqslant 0$ and $m_{rpq}m_{rpq} \geqslant 0$, it follows from (28) that:

$$\tau_{pq}\tau_{pq} \leqslant \frac{1}{\lambda^2}W, \quad m_{rpq}m_{rpq} \leqslant \frac{1}{\mu^2}W \tag{29a, b}$$

in light of which, (27) becomes

$$|f(\varepsilon)| \leqslant 4\pi\varepsilon^2\alpha\beta + (8\pi\varepsilon)^{1/2}\left(\frac{\beta}{\lambda}+\frac{\gamma}{\mu}\right)\left(\int_{C_\varepsilon} W\,\mathrm{d}s\right)^{1/2}. \tag{30}$$

It is time now to use Eq. (19) and write (30) under the final form

$$|f(\varepsilon)| \leqslant \Gamma\varepsilon^2 + \Delta(-\varepsilon f'(\varepsilon))^{1/2}, \tag{31}$$

where $\Gamma = 4\pi\alpha\beta$ and $\Delta = 2\pi^{1/2}(\frac{\beta}{\lambda}+\frac{\gamma}{\mu})$.In the final stage of the proof of the validity of relations (13) and (14), we have to show now that the differential inequality (31) implies the existence of the limit of $f(\varepsilon)$ as $\varepsilon \to +0$ and also that $\lim_{\varepsilon\to+0} f(\varepsilon) = 0$.

The monotonicity of $f(\varepsilon)$ proved earlier assures that either $f(+0)$ exists and is finite or $f(+0) = +\infty$. Thus, there exist the following cases

$$\text{case A}: \ 0 < f(+0) \leqslant \infty, \tag{32a}$$

$$\text{case B}: \ -\infty < f(+0) < 0, \tag{32b}$$

$$\text{case C}: \ f(+0) = 0. \tag{32c}$$

In what follows in this Section, we will show by following the technique of Knowles and Pucik (1973) that the only possibility is case C above.

In case A, it is possible to choose a number $\varepsilon_2 > 0$ small enough to ensure that the inequality (31) holds for $0 < \varepsilon \leqslant \varepsilon_2$ and

$$|f(\varepsilon)| - \Gamma\varepsilon^2 = f(\varepsilon) - \Gamma\varepsilon^2 \geqslant f(\varepsilon) - \Gamma\varepsilon_2^2 \geqslant f(\varepsilon_2) - \Gamma\varepsilon_2^2 > 0. \tag{33}$$

Then, from (31) and (33), it follows that:

$$0 < \left[f(\varepsilon) - \Gamma\varepsilon_2^2\right]^2 \leqslant -\Delta^2\varepsilon f'(\varepsilon) \quad \text{for } 0 < \varepsilon \leqslant \varepsilon_2 \tag{34}$$

and on integrating the latter for $0 < \varepsilon_1 < \varepsilon_2$

$$\Delta^2 \cdot \ln(\varepsilon_2/\varepsilon_1) \leqslant \frac{1}{f(\varepsilon_2) - \Gamma\varepsilon_2^2} - \frac{1}{f(\varepsilon_1) - \Gamma\varepsilon_2^2} \leqslant \frac{1}{f(\varepsilon_2) - \Gamma\varepsilon_2^2} \Rightarrow f(\varepsilon_2) - \Gamma\varepsilon_2^2 \leqslant \frac{\Delta^2}{\ln(\varepsilon_2/\varepsilon_1)}, \tag{35}$$

since, due to the inequalities in (33), $f(\varepsilon_1) - \Gamma\varepsilon_2^2 > 0$. Now, letting $\varepsilon_1 \to +0$ in (35), we get

$$f(\varepsilon_2) - \Gamma\varepsilon_2^2 \leqslant 0, \tag{36}$$

which, however, contradicts the last of the inequalities in (33) and leads to the conclusion that case A is impossible.

In case B, we choose again a number $\varepsilon_2 > 0$ small enough to ensure that the inequality (31) holds for $0 < \varepsilon \leqslant \varepsilon_2$ and, in view of (32b), we obtain

$$|f(\varepsilon)| - \Gamma\varepsilon^2 = -f(\varepsilon) - \Gamma\varepsilon^2 \geqslant -f(\varepsilon) - \Gamma\varepsilon_2^2 \geqslant -f(+0) - \Gamma\varepsilon_2^2 = |f(+0)| - \Gamma\varepsilon_2^2 > 0. \tag{37}$$

From (31) and (37) and by integrating for $0 < \varepsilon_1 < \varepsilon_2$, it results

$$0 < \left[f(\varepsilon) + \Gamma\varepsilon_2^2\right]^2 \leqslant -\Delta^2\varepsilon f'(\varepsilon) \Rightarrow \Delta^2 \cdot \ln(\varepsilon_2/\varepsilon_1) \leqslant \frac{1}{f(\varepsilon_2) + \Gamma\varepsilon_2^2} - \frac{1}{f(\varepsilon_1) + \Gamma\varepsilon_2^2}. \tag{38}$$

But, the first term in the RHS of the latter inequality is negative, so we get

$$-f(\varepsilon_1) - \Gamma\varepsilon_2^2 \leqslant \frac{\Delta^2}{\ln(\varepsilon_2/\varepsilon_1)} \tag{39}$$

and, further, by letting $\varepsilon_1 \to +0$ in (39)

$$|f(+0)| - \Gamma\varepsilon_2^2 \leqslant 0, \tag{40}$$

which, however, contradicts the last of the inequalities in (37) and leads to the conclusion that case B is impossible.

Therefore, we are led necessarily to case C in (32). This means that the limit of the function $f(\varepsilon)$ as $\varepsilon \to +0$ exists indeed and that $\lim_{\varepsilon\to+0} f(\varepsilon) = 0$. Then, by the definition in (17), the existence of $f(+0)$ guarantees the existence of the quantity $\int\int_{D_0} W\,\mathrm{d}A \equiv lim_{\varepsilon\to+0}\int\int_{D_\varepsilon} W\,\mathrm{d}A$ and the validity of (13) and (14).

In conclusion, we have proved that the boundedness requirements in (22) are the necessary conditions for solution uniqueness of the plane crack problem in dipolar gradient elasticity.

4. Uniqueness in the couple-stress theory

In the couple-stress case, Eq. (1) for the strain-energy density is replaced by

$$W \equiv W(\varepsilon_{pq}, \kappa_{pq}), \tag{41}$$

where $\kappa_{pq} = (1/2)e_{qkl}\partial_p\partial_k u_l \equiv e_{qkl}\partial_k\varepsilon_{pl}$ is the so-called torsion–flexure tensor (see e.g. Mindlin and Tiersten, 1962; Muki and Sternberg, 1965) with e_{qkl} being the Levi–Civita permutation symbol. This tensor is the gradient of the rotation or the curl of the strain and it is expressed in dimensions of $[\text{length}]^{-1}$. In terms of the rotation vector $\omega_q = (1/2)e_{qkl}\partial_k u_l$, the torsion-flexure tensor is simply written as $\kappa_{pq} = \partial_p\omega_q$ (recall from Section 2 that $\omega_{pq} = (1/2)(\partial_p u_q - \partial_q u_p)$ is the rotation tensor). Also, one may observe that: (i) the tensor κ_{pq} is asymmetric, and (ii) $\kappa_{pp} = 0$ due to the skew symmetry of the permutation symbol and, therefore, κ_{pq} has only eight independent components. In what follows, we assume the existence of a *positive definite* function $W(\varepsilon_{pq}, \kappa_{pq})$ that has a quadratic form, i.e.

$$W = (1/2)c_{pqlm}\varepsilon_{pq}\varepsilon_{lm} + (1/2)b_{pqlm}\kappa_{pq}\kappa_{lm}, \tag{42}$$

where (c_{pqlm}, b_{pqlm}) are tensors of the material constants.

We proceed now to briefly present the fundamentals of the couple-stress elasticity theory and then to discuss solution uniqueness of the plane crack problem in the context of this theory. Presentations of the basics of the theory can also be found in, e.g., Mindlin and Tiersten (1962), Muki and Sternberg (1965), and Georgiadis and Velgaki (2003). In particular, the latter work provides a more complete version of the theory since it includes the effects of inertia and micro-inertia and reveals the microstructural origin of couple-stress considerations.

The balance laws for the linear and angular momentum for a control volume CV with surface S are written as (Mindlin and Tiersten, 1962; Georgiadis and Velgaki, 2003)

$$\int\int_S t_p^{(n)}\mathrm{d}S + \int\int\int_{\mathrm{CV}} f_p\mathrm{d}(\mathrm{CV}) = 0, \tag{43}$$

$$\int\int_S (x_q t_k^{(n)} e_{pqk} + M_p^{(n)})\mathrm{d}S + \int\int\int_{\mathrm{CV}} \left(x_q\, f_k\, e_{pqk} + C_p\right)\mathrm{d}(\mathrm{CV}) = 0, \tag{44}$$

where $t_p^{(n)}$ is the surface force per unit area (i.e. the traction denoted by the same symbol in Section 2), f_p is the body force per unit volume, $M_p^{(n)}$ is the surface moment per unit area (i.e. the couple traction produced by the double forces $T_{pq}^{(n)}$, which were defined in Section 2), C_p is the body moment per unit volume, and x_q are the components of the position vector of each material particle with elementary volume $\mathrm{d}(CV)$.

Next, pertinent *force-stress* and *couple-stress* tensors are introduced by considering the equilibrium of the elementary material tetrahedron and enforcing (43) and (44), respectively. The force–stress or total stress tensor σ_{pq} (which is asymmetric) is defined by

$$t_p^{(n)} = \sigma_{qp} n_q \tag{45}$$

and the couple-stress tensor μ_{pq} (which is also asymmetric) by

$$M_p^{(n)} = \mu_{qp} n_q. \tag{46}$$

The relations $\mathbf{t}^{(\mathbf{n})} = -\mathbf{t}^{(-\mathbf{n})}$ and $\mathbf{M}^{(\mathbf{n})} = -\mathbf{M}^{(-\mathbf{n})}$, with $\mathbf{n}$ (having direction cosines n_q) being the outward unit vector normal to an area element (on the boundary or on any imagined surface inside the body), can easily be proved to hold by considering the equilibrium of a material 'slice'. The couple-stresses μ_{pq} are expressed in dimensions of [force][length]$^{-1}$. Further, σ_{pq} is decomposed into its symmetric and anti-symmetric part

$$\sigma_{pq} = \tau_{pq} + \alpha_{pq} \tag{47}$$

with $\tau_{pq} = \tau_{qp}$ and $\alpha_{pq} = -\alpha_{qp}$, whereas μ_{pq} is decomposed into its deviatoric $\mu_{pq}^{(\mathrm{D})}$ and spherical $\mu_{pq}^{(\mathrm{S})}$ part in the following manner:

$$\mu_{pq} = m_{pq} + (1/3)\delta_{pq}\mu_{kk}, \tag{48}$$

where $m_{pq} \equiv \mu_{pq}^{(\mathrm{D})}$, $\mu_{pq}^{(\mathrm{S})} \equiv (1/3)\delta_{pq}\mu_{kk}$, and δ_{pq} is the Kronecker delta.

The couple-stress tensor μ_{pq} is related with the double-stress tensor m_{rpq} defined in Section 2 through the equation $\mu_{rl} = (1/2)e_{lpq}m_{r[pq]}$, where $m_{r[pq]} \equiv (1/2)(m_{rpq} - m_{rqp})$.

Now, in view of the above definitions and by employing the divergence theorem, the following equations of force and moment equilibrium are obtained:

$$\partial_p\sigma_{pq} + f_q = 0, \tag{49}$$

$$\partial_p\mu_{pq} + \sigma_{kp}\, e_{pqk} + C_q = 0 \tag{50}$$

or, by virtue of (47)

$$\partial_p\tau_{pq} + \partial_p\alpha_{pq} + f_q = 0, \tag{51}$$

$$(1/2)\partial_p\mu_{pk}e_{pqk} + \alpha_{pq} + (1/2)C_k e_{pqk} = 0. \tag{52}$$

Further, combining (51) and (52) and also taking into account that $\mathrm{curl}(\mathrm{div}((1/3)\delta_{pq}\mu_{kk})) = 0$ yields the *single* equation of equilibrium

$$\partial_p\tau_{pq} - (1/2)\partial_p\partial_r m_{rk}e_{pqk} + f_q - (1/2)\partial_p C_k e_{pqk} = 0. \tag{53}$$

Linear constitutive relations can be derived now from (42) and the usual variational considerations

$$\tau_{pq} \equiv \frac{\partial W}{\partial\varepsilon_{pq}} = c_{pqlm}\varepsilon_{lm}, \qquad m_{pq} \equiv \frac{\partial W}{\partial\kappa_{pq}} = b_{pqlm}\kappa_{lm}. \tag{54a, b}$$

As in Section 2 and Section 3, due to the linearity of the constitutive relations, we can appropriately write $W \equiv W(\tau_{pq}, m_{pq})$.

Finally, the following points are of notice: (i) since $\kappa_{pp} = 0$ holds true, $m_{pp} = 0$ is also valid and therefore the tensor m_{pq} has only eight independent components. (ii) The scalar $(1/3)\mu_{kk}$ of the couple-stress tensor μ_{pq} does not appear in the final equation of motion and in the constitutive equations either. Consequently, $(1/3)\mu_{kk}$ is left *indeterminate* within the couple-stress theory. In other words, the field μ_{pq} is unique except for an arbitrary additive (constant) isotropic couple-stress field.

We proceed now to discuss the uniqueness of the plane crack problem in the context of the theory presented above. We consider the same configuration of a cracked plane domain as that in Section 3 and follow essentially the same procedure. In the present case, the boundary value problem is stated as follows: determine the solution $(u_q, \omega_q, \tau_{pq}, m_{pq})$ to the problem described by Eqs. (53) and (54) holding on D_0, by the general conditions (45) and (46) holding along any boundary, and under given tractions t_q^* and M_q^* on the boundary C and stress-free boundary conditions on L_0.

More specifically, the latter conditions on L_0 are written in, e.g., plane-strain as

$$\sigma_{22} = 0, \quad \sigma_{21} = 0, \tag{55a, b}$$

$$m_{23} = 0. \tag{56}$$

Notice further that except for ω_3 and (κ_{13}, κ_{23}) all other components of the rotation vector and the torsion-flexure tensor identically vanish, in the plane-strain case considered here. The non-vanishing components (τ_{11},τ_{12}, τ_{22}) and (m_{13},m_{23}) follow from (54). Then, (α_{11},α_{12},α_{21},α_{22}) are found from (52) and (σ_{11},σ_{12},σ_{21},σ_{22}) are provided by (47).

Finally, just as in the case of dipolar gradient theory treated earlier, we also state the appropriate smoothness conditions that the solution to the above boundary value problem should obey. These conditions are written as

$$u_q \in \mathbf{C}^1(D^+) \cap \mathbf{C}^1(D^-) \cap \mathbf{C}^4(D_0), \tag{57a}$$

$$\omega_q \in \mathbf{C}(D^+) \cap \mathbf{C}(D^-) \cap \mathbf{C}^3(D_0), \tag{57b}$$

$$\tau_{pq} \in \mathbf{C}(D^+) \cap \mathbf{C}(D^-) \cap \mathbf{C}^1(D_0), \tag{57c}$$

$$m_{pq} \in \mathbf{C}(D^+) \cap \mathbf{C}(D^-) \cap \mathbf{C}^2(D_0). \tag{57d}$$

Starting our uniqueness considerations, we first write the Principle of Virtual Work in the couple-stress elasticity (the operator δ( ) below denotes weak variations)

$$\int\int_S (t_q^{(n)}\delta u_q + M_q^{(n)}\delta\omega_q)\mathrm{d}S + \int\int\int_{\mathrm{CV}} (f_q\delta u_q + C_q\delta\omega_q)\mathrm{d}(\mathrm{CV})$$
$$= \int\int\int_{\mathrm{CV}} (\tau_{pq}\delta\varepsilon_{pq} + m_{pq}\delta\kappa_{pq})\mathrm{d}(\mathrm{CV}) \tag{58}$$

and then relations analogous to the ones in (13) and (14)

$$\int\int_{D_0} W\,\mathrm{d}A < \infty, \tag{59}$$

$$2\int\int_{D_0} W\,\mathrm{d}A = \int\int_{D_0} f_q u_q\,\mathrm{d}A + \int\int_{D_0} C_q\omega_q\,\mathrm{d}A + \int_C t_q^* u_q\,\mathrm{d}s + \int_C M_q^*\omega_q\,\mathrm{d}s, \tag{60}$$

where pertinent definitions of the domains and contours are given in Section 3. Clearly, (60) derives from the general variational form in (58). Notice that f_q and u_q have components, respectively, (f_1,f_2) and (u_1,u_2) only, whereas C_q and ω_q have the components, respectively, C_3 and ω_3 only. Also, both fields f_q and C_q are assumed to be continuous.

As in the previous case of Section 3, once the relations (59) and (60) are established, uniqueness follows as in the standard Kirchhoff theorem. In other words, these two relations are necessary for the extension of a Kirchhoff-type theorem in the plane crack problem stated before. A uniqueness theorem in the couple-stress elasticity that is analogous to Kirchhoff's theorem was proved indeed by Mindlin and Tiersten (1962). Thus, in what follows, we will concentrate on determining the conditions on the fields (f_q,C_q,u_q,ω_q) that guarantee the validity of relations (59) and (60).

The starting point is again proving the validity of the following equation:

$$2\int\int_{D_0} W\,\mathrm{d}A = \int\int_{D_0} f_q u_q\,\mathrm{d}A + \int\int_{D_0} C_q\omega_q\,\mathrm{d}A + \int_C t_q^* u_q\,\mathrm{d}s + \int_C M_q^*\omega_q\,\mathrm{d}s + \int_{C_\varepsilon} t_q u_q\,\mathrm{d}s + \int_{C_\varepsilon} M_q\omega_q\,\mathrm{d}s, \tag{61}$$

where (t_q^*, M_q^*) are the given (known) tractions on the boundary C, whereas $(t_q \equiv t_q^{(n)}, M_q \equiv M_q^{(n)})$ are unknown tractions along C_ε. The latter tractions are related, of course, with the stresses through (45) and (46). The proof of (61) is accomplished as the one of the analogous equation (15) in Section 3, i.e. through the use of boundary conditions (local equilibrium), equations of equilibrium (global equilibrium), and the divergence theorem.

Then, for all positive and sufficiently small numbers ε, we define a function $g(\varepsilon)$ by

$$g(\varepsilon) \equiv 2\int\int_{D_\varepsilon} W\,\mathrm{d}A - \int\int_{D_0} f_q u_q\,\mathrm{d}A - \int\int_{D_0} C_q\omega_q\,\mathrm{d}A - \int_C t_q^* u_q\,\mathrm{d}s - \int_C M_q^*\omega_q\,\mathrm{d}s \tag{62}$$

aiming at showing that $\lim_{\varepsilon\to+0} g(\varepsilon) = 0$ and, accordingly, that $\int\int_{D_0} W\,\mathrm{d}A \equiv \lim_{\varepsilon\to+0}\int\int_{D_\varepsilon} W\,\mathrm{d}A < \infty$. From (62), through a Leibnitz-type differentiation and by taking into account the positive definiteness of W, $g(\varepsilon)$ is proved to be a monotone non-increasing function, i.e. $g'(\varepsilon) = -2\int_{C_\varepsilon} W\,\mathrm{d}s \leqslant 0$. Also, under the definition in (62), we can write (61) in the form

$$g(\varepsilon) = -\int\int_{G_\varepsilon} f_q u_q\,\mathrm{d}A - \int\int_{G_\varepsilon} C_q\omega_q\,\mathrm{d}A + \int_{C_\varepsilon} t_q u_q\,\mathrm{d}s + \int_{C_\varepsilon} M_q\omega_q\,\mathrm{d}s. \tag{63}$$

Next, we work on the last two integrals of the RHS of (63) and exploit the fact that neither the anti-symmetric part of the force-stress nor the spherical part of the couple-stress contributes to the strain-energy density in the couple-stress theory (cf. Eq. (54) and the exposition in Mindlin and Tiersten (1962)) obtaining

$$\int_{C_\varepsilon} t_q u_q\,\mathrm{d}s + \int_{C_\varepsilon} M_q\omega_q\,\mathrm{d}s = \int_{C_\varepsilon} n_p(\sigma_{pq}u_q + \mu_{pq}\omega_q)\mathrm{d}s = \int_{C_\varepsilon} n_p(\tau_{pq}u_q + m_{pq}\omega_q)\mathrm{d}s. \tag{64}$$

Therefore, (64) leads us to write (63) in the form

$$g(\varepsilon) = -\int\int_{G_\varepsilon} f_q u_q\,\mathrm{d}A - \int\int_{G_\varepsilon} C_q\omega_q\,\mathrm{d}A + \int_{C_\varepsilon} n_p\tau_{pq}u_q\,\mathrm{d}s + \int_{C_\varepsilon} n_p m_{pq}\omega_q\,\mathrm{d}s. \tag{65}$$

Working now as in Section 3 and employing the triangle and the Cauchy–Schwarz inequalities, we obtain the following bound for $g(\varepsilon)$

$$|g(\varepsilon)| \leqslant 4\pi\varepsilon^2\alpha\beta + 4\pi\varepsilon^2\eta\zeta + \beta(8\pi\varepsilon)^{1/2}\left(\int_{C_\varepsilon} \tau_{pq}\tau_{pq}\,\mathrm{d}s\right)^{1/2} + \zeta(8\pi\varepsilon)^{1/2}\left(\int_{C_\varepsilon} m_{pq}m_{pq}\,\mathrm{d}s\right)^{1/2}, \tag{66}$$

where the positive constants $(\alpha, \beta, \eta, \zeta)$ are the bounds of the pertinent field quantities (body force, body couple, displacement, and rotation), i.e.

$$|f_q| \leqslant \alpha \quad \forall x_p \in D_0, \tag{67a}$$

$$|C_q| \leqslant \eta \quad \forall x_p \in D_0, \tag{67b}$$

$$|u_q| \leqslant \beta \quad \forall x_p \in D_0, \tag{67c}$$

$$|\omega_q| \leqslant \zeta \quad \forall x_p \in D_0. \tag{67d}$$

The above restrictions are the necessary edge conditions for the fields near to the crack tips that guarantee a unique solution to the plane crack problem in the case of the couple-stress theory.

Next, we turn again to the strain-energy density and make use of the fact that this is a quadratic function of its arguments. Consequently, there exist positive constants λ and μ such that

$$W(\tau_{pq}, m_{pq}) \geqslant \lambda^2 \tau_{pq}\tau_{pq} + \mu^2 m_{pq} m_{pq} \tag{68}$$

and since $\tau_{pq}\tau_{pq} \geqslant 0$ and $m_{pq}m_{pq} \geqslant 0$, it follows that:

$$\tau_{pq}\tau_{pq} \leqslant \frac{1}{\lambda^2} W, \quad m_{pq}m_{pq} \leqslant \frac{1}{\mu^2} W. \tag{69a, b}$$

Employing finally (69) and the expression $g'(\varepsilon) = -2\int_{C_\varepsilon} W \, \mathrm{d}s$, (66) takes the form

$$|g(\varepsilon)| \leqslant \Phi\varepsilon^2 + \Psi(-\varepsilon g'(\varepsilon))^{1/2}, \tag{70}$$

where $\Phi = 4\pi(\alpha\beta + \eta\zeta)$ and $\Psi = 2\pi^{1/2}(\frac{\beta}{\lambda} + \frac{\zeta}{\mu})$. Since now (70) is identical in form with (31), the rest of the proof follows the same lines as in Section 3.

In conclusion, the boundedness requirements in (67) are the necessary conditions for solution uniqueness of the plane crack problem in couple-stress elasticity.

5. Uniqueness in the pure strain-gradient theory

In the pure strain-gradient case, Eq. (1) for the strain-energy density is considered but now with κ_{rpq} being the gradient of only the strain field (and not of the displacement-gradient field), i.e. $\kappa_{rpq} = \partial_r \varepsilon_{pq}$. This is form II in Mindlin's (1964) paper. Obviously, it is $\kappa_{rpq} \equiv \kappa_{rqp}$. Stresses are defined as in (3) and, accordingly, the dipolar stress tensor exhibits the latter type of symmetry, i.e. $m_{rpq} \equiv m_{rqp}$.

This formulation of the dipolar gradient theory does not take into consideration rotation gradients. All governing equations pertaining to form I (presented in Section 2) and the uniqueness considerations (presented in Section 3) are also valid for form II provided that the proper symmetries for all tensors are followed. In this way, we end up with the following edge conditions, which guarantee uniqueness of the plane crack problem within the pure strain-gradient theory of elasticity

$$|f_q| \leqslant \alpha \quad \forall x_p \in D_0, \tag{71a}$$

$$|u_q| \leqslant \beta \quad \forall x_p \in D_0, \tag{71b}$$

$$|\varepsilon_{pq}| \leqslant \xi \quad \forall x_p \in D_0, \tag{71c}$$

where (α, β, ξ) are positive constants.

6. Concluding remarks

In this paper, we derived the pertinent edge conditions (i.e. boundedness requirements for certain fields in the vicinity of crack tips), which guarantee solution uniqueness of plane crack problems in dipolar gradient elasticity and in couple-stress elasticity. The derivation was based on energy related arguments. It was shown that more stringent conditions are required for these generalized continuum theories as compared with the ones required for standard elasticity. The information provided by edge conditions can be useful in the course of solving (by either analytical or numerical techniques) boundary value problems involving cracks, since one may know in advance the behavior that certain field quantities (displacement gradient or rotation or strain) should exhibit in the crack-tip region and thus may check upon the appropriateness of candidate solutions.

We should also notice that the form of the constitutive relations *does not* affect the boundedness requirements as long as the constitutive relations remain linear. In view of this observation, the present results apply also to a recent modification (Yang et al., 2002; Lam et al., 2003) of the general theory. This modification is based on postulating an additional equilibrium condition (to the standard balance of linear and angular momenta)—the equilibrium of moments of couples. The latter postulate does not derive from general principles of mechanics and results in a reduced number of material constants (as compared with the number of material constants existing in the general theory). As indicated by, among others, Jaunzemis (1967) and Mindlin and Eshel (1968), the principles of linear and angular momenta *suffice* to derive well-posed dipolar theories. Considering now the modification by Yang et al. (2002) in our problem, one may observe that since the rotation vector ω_q enters Eqs. (61) and (64) of the present paper and the form of the Principle of Virtual Work is identical in our work (Eq. (58)) with that in the work by Yang et al. (their Eqs. (34) and (35)), the boundedness requirement on ω_q near the crack tips still *remains* a necessary one for the specific case of Yang et al. (2002). Clearly, the same boundedness requirements apply also for the simplified theory because the modification has to do with the particular form of the constitutive relations and not with the basic variational formulation. The latter one remains the same in the simplified case. Finally, one may observe that the particular form (i.e. the number of material constants) of the linear constitutive relations does not affect the boundedness requirements. Indeed, Eq. (68) of the present paper remains valid in *both* cases of an asymmetric couple-stress tensor m_{pq} (general couple-stress theory) or a symmetric one (simplified case of Yang et al., 2002).

In closing, we should notice that a host of existing solutions to crack problems (solutions employing generalized continuum theories) clearly provides a corroboration of the present results (i.e. the near-tip additional conditions to those given by classical elasticity). In particular, the condition of boundedness of the displacement-gradient field $u_{q,p}$ is fulfilled by the solutions of Shi et al. (2000) and Georgiadis (2003) in the case of gradient theory, whereas the condition of boundedness of the rotation field ω_q is fulfilled by the solutions of Sternberg and Muki (1967), Atkinson and Leppington (1977), Huang et al. (1997) and Huang et al. (1999) in the case of couple-stress theory.

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Appendix A

We prove here a theorem of uniqueness of the Kirchhoff type (or Kirchhoff–Neumann type) within the dipolar gradient elasticity theory and under the requirement of a positive definite strain-energy density.

First, we write the Principle of Virtual Work for this theory (Mindlin, 1964; Georgiadis and Grentzelou, in preparation)

$$\int\int_S [P_q \delta u_q + R_q D(\delta u_q)] \mathrm{d}S + \int\int\int_{\mathrm{CV}} f_q \delta u_q \mathrm{d}(\mathrm{CV}) = \int\int\int_{\mathrm{CV}} (\tau_{pq}\delta\varepsilon_{pq} + m_{rpq}\delta\kappa_{rpq}) \mathrm{d}(\mathrm{CV}), \tag{A.1}$$

where all symbols are defined in the main body of the paper and the operator δ( ) denotes weak variations.

We will prove solution uniqueness by *reductio ad absurdum*. To this end, we assume that two *different* solutions do exist for the same problem (i.e. same material, geometry, boundary conditions and body forces), say $\Lambda' = (u'_q, Du'_q, \tau'_{pq}, m'_{rpq})$ and $\Lambda'' = (u''_q, Du''_q, \tau''_{pq}, m''_{rpq})$.

Then, due to the assumed identical boundary conditions, we write

$$\int\int_S \left[\left(P'_q - P''_q\right)\left(u'_q - u''_q\right) + \left(R'_q - R''_q\right)\left(Du'_q - Du''_q\right)\right] dS = 0 \tag{A.2}$$

In addition, due to the linearity of the governing equations, the difference of the two solutions defined as $\Lambda = (u_q, Du_q, \tau_{pq}, m_{rpq})$, where $u_q = u'_q - u''_q$, $Du_q = Du'_q - Du''_q$, $\tau_{pq} = \tau'_{pq} - \tau''_{pq}$ and $m_{rpq} = m'_{rpq} - m''_{rpq}$, will satisfy Eqs. (4)–(7) with $f_q = f'_q - f''_q \equiv 0$, $P_q = P'_q - P''_q \equiv 0$ and $R_q = R'_q - R''_q \equiv 0$. Since now the LHS of (A.1) vanishes, due to (A.2) and the fact that $f_q \equiv 0$, we are led to the conclusion that the RHS of (A.1) vanishes too, which, however, is not true because the strain energy was assumed to be a *positive definite* quantity.

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